Government College of Engineering and Research
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Fundamental of Finite Element Analysis

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Unit 4
1D Steady State heat Transfer Problem

Unit Outcomes :

After completion of this unit student should be able to

1. Derive elemental stiffness matrix and load vector for one dimensional steady state heat transfer problem.
2. Explain the boundary condition for heat transfer problem.
3. Solve the numerical on composite wall and thin fin problem.

Learning Outcomes

- After completion of this lecture student should be able to

1) Derivation of elemental stiffness matrix and load vector for one dimensional Heat transfer problem.

(i) **Conduction** : It can take place due to molecular interaction or movement of free electrons.

Example : Fins on automobile cylinder.

One dimensional heat flow is governed by Fourier's law and is given as ;

$$Q_x = -K\frac{\partial T}{\partial x}$$

Where, Q_x = heat flux in x-direction; K = thermal conductivity

T = temperature; $\frac{\partial T}{\partial x}$ = temperature gradient along x-direction

Negative sign indicates that heat is flowing in the decreasing direction of temperature.

(ii) Convection :

- It takes place in a fluid medium.
- Fluid motion take place w.r.t. surface.
- When fluid motion takes place due to buoyancy effects obtained from the density variation caused by the temperature difference in the fluid, it is known as free convection.
- When fluid motion takes place by a blower or fan, it is known as forced convection.
- Heat convection is given by ;

$$q = h (T_S - T_\infty)$$

Where,
q = heat flux at the surface
h = convective heat transfer
T_s = surface temperature
T_∞ = fluid temperature

Heat Conduction matrix for 1D Steady State heat Transfer

- Governing equation for heat transfer through composite wall is given by ,

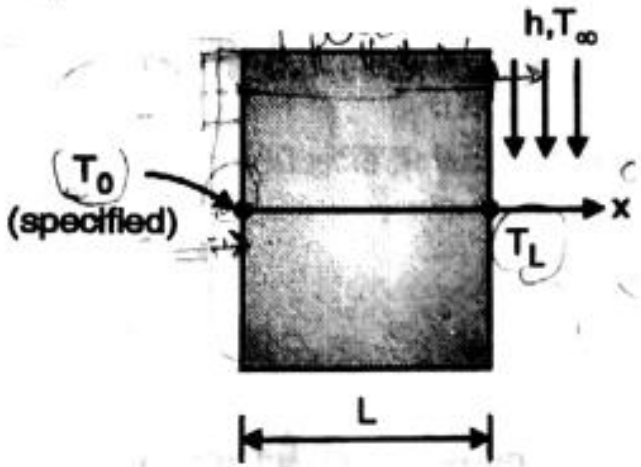
$$\frac{Kd^2T}{dx^2} + \dot{q} = 0$$

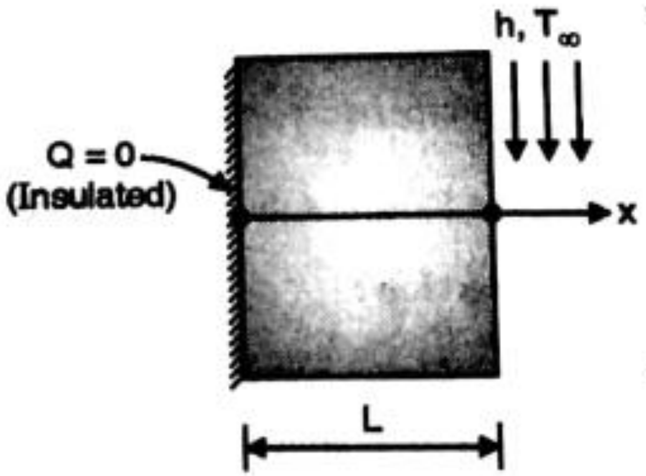
Where,

K = Thermal conductivity

T = Temperature at any distance

$\dot{q}$ = Internal heat generated/unit vol.





Shape function of 1D element

- Finite element modelling is shown in below Fig. 5.3.2.

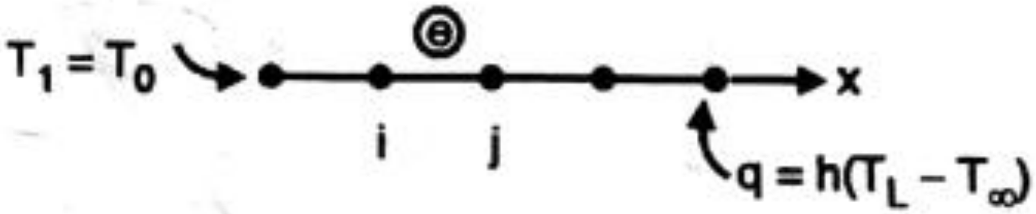


Fig. 5.3.2

The temperature polynomial is given by,

$$[T] = [N]\{T_e\}$$

For one dimensional two noded element the shape function is given by,

$T = a_1 + a_2 x$. (2 degree of freedom
temp. increase of degree)

$$T = [1 \quad x] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

at $x = 0 \quad T = T_1$

$x = L \quad T = T_2$.

max at $x = 0$

$T = a_1 \quad \therefore \quad a_1 = T_1$

at $x = L \quad T = T_2$.

$T = T_1 + a_2 L$.

$T_2 = T_1 + a_2 L$

$$\frac{T_2 - T_1}{L} = a_2.$$

$$\therefore T = \left(1 - \frac{x}{L}\right) T_1 + \frac{x}{L} T_2.$$

shape function.

$$\frac{dT}{dx} = \frac{d}{dx}\left[\left(1 - \frac{x}{L}\right)T_1 + \frac{x}{L}T_2\right]$$

$$= -\frac{dT_1}{L\,dx} + \frac{dT_2}{L\,dx}$$

$$= \frac{1}{L}\left[-1 \quad 1\right]$$

$$= -\frac{1}{L}T_1 + \frac{1}{L}\times T_2$$

$$= \left[-\frac{1}{L} \quad \frac{1}{L}\right]\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

Using Galerkin Method

By using Galerkin method ;

$$\int_0^1 W\,R(x)\,dx = 0$$

Where, W = weighting factor

R (x) = error function

$$\int_0^l W\left[\frac{Kd^2T}{dx^2}+\dot{q}\right] dx = 0$$

$$\int_0^l \left(W\left[\frac{Kd^2T}{dx^2}\right]+W\dot{q}\right) dx = 0$$

$$\int_0^l WK\frac{d^2T}{dx^2}\,dx + \int_0^l W\,\dot{q}\,dx = 0$$

Using the formulae given below the above integral is solved

$$\int udv = uv - \int vdu$$

$$\int d(uv) = \int udv + \int vdu$$

$$\left[WK\frac{dT}{dx}\right]_0^l - \int_0^l K\frac{dW}{dx}\frac{dT}{dx}\,dx + \int_0^l W\,\dot{q}\,dx = 0$$

$$\left[WK\frac{dT}{dx}\right]_0^l + \int_0^l W\,\dot{q}\,dx = \int_0^l K\frac{dW}{dx}\frac{dT}{dx}\,dx$$

$$\int_0^l \left(W\left[\frac{Kd^2T}{dx^2}\right] + W\dot{q} \right) dx = 0$$

$$\int_0^l WK\frac{d^2T}{dx^2}\,dx + \int_0^l W\,\dot{q}\,dx = 0$$

Using the formulae given below the above integral is solved

$$\int u\,dv = uv - \int v\,du$$

$$\int d(uv) = \int u\,dv + \int v\,du$$

$$\left[WK\frac{dT}{dx}\right]_0^l - \int_0^l K\frac{dW}{dx}\frac{dT}{dx}\,dx + \int_0^l W\,\dot{q}\,dx = 0$$

$$\left[WK\frac{dT}{dx}\right]_0^l + \int_0^l W\,\dot{q}\,dx = \int_0^l K\frac{dW}{dx}\frac{dT}{dx}\,dx$$

A

B C

$$T = \left(1-\frac{x}{l}\right)T_1 + \frac{x}{l}\,T_2$$

$$\therefore\ W_1 = 1-\frac{x}{l}\ ,\quad W_2 = \frac{x}{l}$$

$$\frac{dW_1}{dx} = \frac{-1}{l};\qquad \frac{dW_2}{dx} = \frac{1}{l}$$

A

$$= \int_0^l K\frac{dW}{dx}\frac{dT}{dx}\,dx$$

$$= \left(K\frac{dW}{dx}\frac{dT}{dx}\cdot x\right)_0^l$$

$$= K\left(\frac{dW}{dx}\right)_0^l\left(\frac{dT}{dx}\right)_0^l(x)_0^l$$

$$= K\begin{bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{bmatrix}\frac{1}{l}[-1,\ 1]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} l$$

By putting value of dW/dx and dT/dx

$$= K \cdot \frac{1}{l}\begin{bmatrix} -1 \\ 1 \end{bmatrix}[-1\ \ 1]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

A $$= \frac{K}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

B $$= \left[WK\frac{dT}{dx} \right]_0^l$$

$$= \left[\left(1 - \frac{x}{L}\right) K\frac{dT}{dx} + \frac{x}{L} K\frac{dT}{dx} \right]_{at\ x=L} -$$

$$\left[\left(1 - \frac{x}{L}\right) K\frac{dT}{dx} + \frac{x}{L} K\frac{dT}{dx} \right]_{at\ x=0}$$

$$K\frac{dT}{dx}\Big|_{at\ L} - K\frac{dT}{dx}\Big|_{at\ x=0}$$

$$\begin{bmatrix} -Q_1 \\ Q_2 \end{bmatrix}$$

C

$$= \int_0^l W.\dot{q}\,dx.$$

$$= \dot{q} \int_0^l \left(\left(1 - \frac{x}{l}\right) + \frac{x}{l} \right) dx$$

$$= \dot{q} \left[x - \frac{x^2}{2l} \right]_0^l + \left[\frac{x^2}{2l} \right]_0^l$$

$$= \dot{q} \left[l - \frac{l^2}{2l} \right]_0^l + \dot{q} \left[\frac{l^2}{2l} \right]_0^l$$

$$= \dot{q} \left[l/2 \right] + \dot{q} \left[l/2 \right]$$

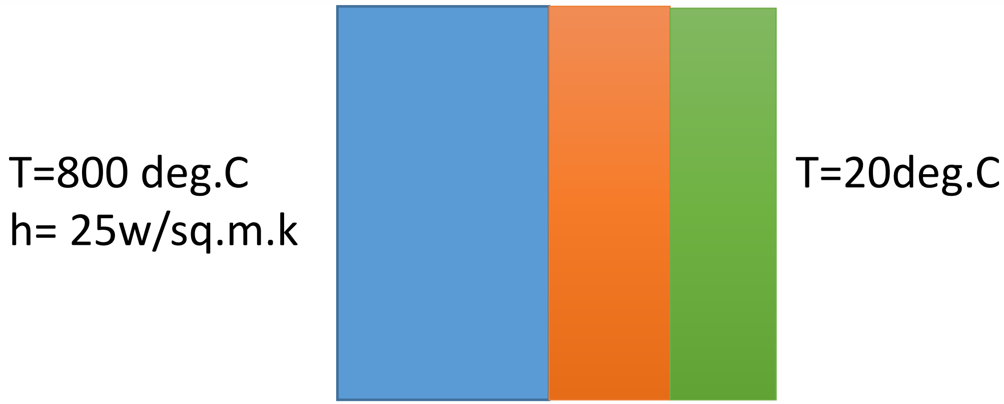
$$= \dot{q} \begin{bmatrix} l/2 \\ l/2 \end{bmatrix}$$

$$\frac{K}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \dot{q} \begin{Bmatrix} l/2 \\ l/2 \end{Bmatrix} + \begin{Bmatrix} -Q_1 \\ Q_2 \end{Bmatrix}$$

A composite slab consists of three materials with thermal conductivities of 20 W/m °K, 30 W/m °K, 50 W/m °K and thicknesses 0.3 m, 0.15 m and 0.15 m respectively. The outer surface is at 20 °C and the inner surface is exposed to the convective heat transfer coefficient of 25 W/m^2 °K and a medium at 800 °C. Determine the temperature distribution within the wall.

Solution

Since the plate can be considered infinite, heat transfer can be assumed to be one-dimensional across the thickness. Heat conduction matrices of the three elements covering the three materials are obtained as



The plane wall shown in Figure 13–13 is 1 m thick. The left surface of the wall ($x = 0$) is maintained at a constant temperature of 200 °C, and the right surface ($x = L = 1$ m) is insulated. The thermal conductivity is $K_{xx} = 25$ W/(m · °C) and there is a uniform generation of heat inside the wall of $Q = 400$ W/m^3. Determine the temperature distribution through the wall thickness.

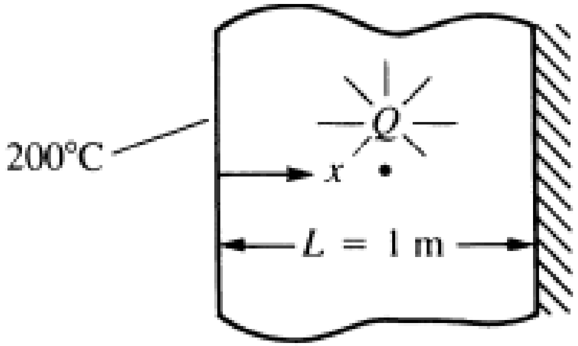


Figure 13–13 Conduction in a plane wall subjected to uniform heat generation

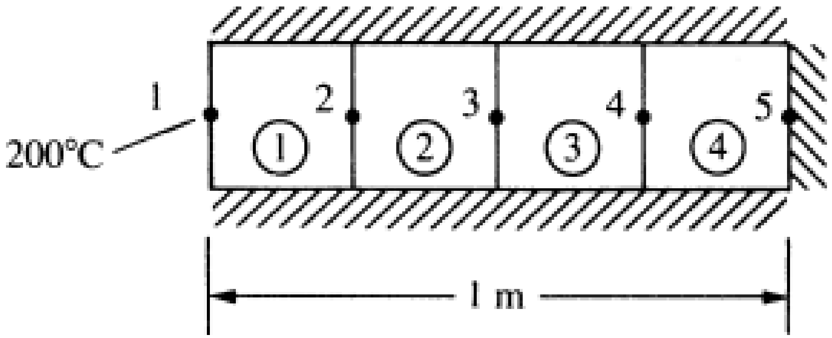


Figure 13–14 Discretized model of Figure 13–13

This problem is assumed to be approximated as a one-dimensional heat-transfer problem. The discretized model of the wall is shown in Figure 13–14. For simplicity, we use four equal-length elements all with unit cross-sectional area ($A = 1\ \text{m}^2$). The unit area represents a typical cross section of the wall. The perimeter of the wall model is then insulated to obtain the correct conditions.

Using Eqs. (13.4.22) and (13.4.28), we calculate the element stiffness matrices as follows:

$$\frac{AK_{xx}}{L} = \frac{(1\ \text{m}^2)[25\ \text{W}/(\text{m}\cdot{}^\circ\text{C})]}{0.25\ \text{m}} = 100\ \text{W}/{}^\circ\text{C}$$

For each identical element, we have

$$[k] = 100\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ \text{W}/{}^\circ\text{C} \qquad (13.4.54)$$

Because no convection occurs, h is equal to zero; therefore, there is no convection contribution to $\underline{k}$.

The element force matrices are given by Eq. (13.4.26). With $Q = 400\ \text{W/m}^3$, $q = 0$, and $h = 0$, Eq. (13.4.26) becomes

$$\{f\} = \frac{QAL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{13.4.55}$$

Evaluating Eq. (13.4.55) for a typical element, such as element 1, we obtain

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \frac{(400\ \text{W/m}^3)(1\ \text{m}^2)(0.25\ \text{m})}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 50 \\ 50 \end{Bmatrix}\ \text{W} \tag{13.4.56}$$

The force matrices for all other elements are equal to Eq. (13.4.56).

The assemblage of the element matrices, Eqs. (13.4.54) and (13.4.56) and the other force matrices similar to Eq. (13.4.56), yields

$$100\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 50 \\ 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix} + \begin{Bmatrix} Q1 \\ -Q2+Q2 \\ -Q3+Q3 \\ -Q4+Q4 \\ Q5 \end{Bmatrix}$$

t1= 200; Q1 =unknown; Q5=0

$$100\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 200 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 50 \\ 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix} + \begin{Bmatrix} -Q1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

By using penalty approach

$$100\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 200 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 50 \\ 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix} + \begin{Bmatrix} -Q1 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$100\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 100 \\ 100 \\ 50 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} -200 \begin{Bmatrix} -100 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$t_2 = 203.5\,^{\circ}C$ $t_3 = 206\,^{\circ}C$ $t_4 = 207.5\,^{\circ}C$ $t_5 = 208\,^{\circ}C$

Using the first of Eqs. yields the rate of heat flow out the left end:

$$Q1 = 100(t_1 - t_2) - 50$$

$$= 100(200 - 203.5) - 50$$

$$Q1 = -400 \text{ W}$$

Thank You

For Your Attention